Progressions for the Common Core State Standards in Mathematics (draft)*

12 August 2011

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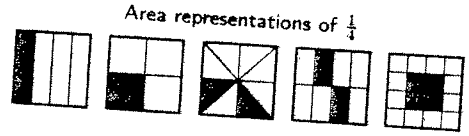
Initially, students can use an intuitive notion of congruence ("same size and same shape") to explain why the parts are equal, e.g., when they divide a square into four equal squares or four equal rectangles.

Students come to understand a more precise meaning for "equal parts" as "parts with equal measurements." For example, when a ruler is partitioned into halves or quarters of an inch, they see that each subdivision has the same length. In area models they reason about the area of a shaded region to decide what fraction of the whole it represents (MP3).

Area representations of $\frac{1}{4}$



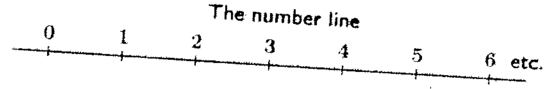
In each representation the square is the whole. The two squares on the left are divided into four parts that have the same size and shape, and so the same area. In the three squares on the right, the shaded area is $\frac{1}{4}$ of the whole area, even though it is not easily seen as one part in a division of the square into four parts of the same shape and size.

The goal is for students to see unit fractions as the basic building blocks of fractions, in the same sense that the number 1 is the basic building block of the whole numbers; just as every whole number is obtained by combining a sufficient number of 1s, every fraction is obtained by combining a sufficient number of unit fractions.

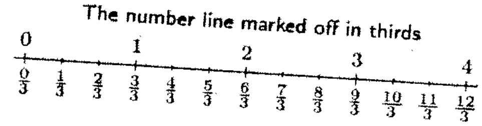
The number line and number line diagrams On the number line, the whole is the *unit interval*, that is, the interval from 0 to 1, measured by length. Iterating this whole to the right marks off the whole numbers, so that the intervals between consecutive whole numbers, from 0 to 1, 1 to 2, 2 to 3, etc., are all of the same length, as shown. Students might think of the number line as an infinite ruler.

To construct a unit fraction on a number line diagram, e.g. $\frac{1}{3}$, students partition the unit interval into 3 intervals of equal length and recognize that each has length $\frac{1}{3}$. They locate the number $\frac{1}{3}$ on the number line by marking off this length from 0, and locate other fractions with denominator 3 by marking off the number of lengths indicated by the numerator.[3.NF.2]

The number line



The number line marked off in thirds



[3.NF.2] Understand a fraction as a number on the number line; represent fractions on a number line diagram.

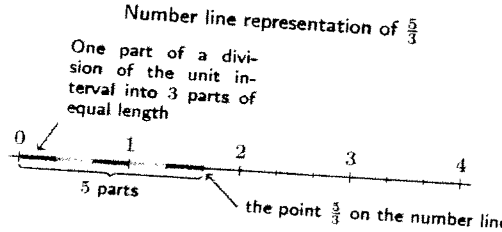
a Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.

b Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

Students sometimes have difficulty perceiving the unit on a number line diagram. When locating a fraction on a number line diagram, they might use as the unit the entire portion of the number line that is shown on the diagram, for example indicating the number 3 when asked to show $\frac{3}{4}$ on a number line diagram marked from 0 to 4. Although number line diagrams are important representations for students as they develop an understanding of a fraction as a number, in the early stages of the NF Progression they use other representations such as area models, tape diagrams, and strips of paper. These, like number line diagrams, can be subdivided, representing an important aspect of fractions.

The number line reinforces the analogy between fractions and whole numbers. Just as 5 is the point on the number line reached by marking off 5 times the length of the unit interval from 0, so $\frac{5}{3}$ is the point obtained in the same way using a different interval as the basic unit of length, namely the interval from 0 to $\frac{1}{3}$.

Number line representation of $\frac{5}{3}$



Equivalent fractions Grade 3 students do some preliminary reasoning about equivalent fractions, in preparation for work in Grade 4. As students experiment on number line diagrams they discover that many fractions label the same point on the number line, and are

Grade 4

Grade 4 students learn a fundamental property of equivalent fractions: multiplying the numerator and denominator of a fraction by the same non-zero whole number results in a fraction that represents the same number as the original fraction. This property forms the basis for much of their other work in Grade 4, including the comparison, addition, and subtraction of fractions and the introduction of finite decimals.

Equivalent fractions Students can use area models and number line diagrams to reason about equivalence.[4.NF.1] They see that the numerical process of multiplying the numerator and denominator of a fraction by the same number, n, corresponds physically to partitioning each unit fraction piece into n smaller equal pieces. The whole is then partitioned into n times as many pieces, and there are n times as many smaller unit fraction pieces as in the original fraction.

This argument, once understood for a range of examples, can be seen as a general argument, working directly from the Grade 3 understanding of a fraction as a point on the number line.

The fundamental property can be presented in terms of division, as in, e.g.

$$\frac{28}{36} = \frac{28 \div 4}{36 \div 4} = \frac{7}{9}.$$

Because the equations $28 \div 4 = 7$ and $36 \div 4 = 9$ tell us that $28 = 4 \times 7$ and $36 = 4 \times 9$, this is the fundamental fact in disguise:

$$\frac{4 \times 7}{4 \times 9} = \frac{7}{9}.$$

It is possible to over-emphasize the importance of simplifying fractions in this way. There is no mathematical reason why fractions must be written in simplified form, although it may be convenient to do so in some cases.

Grade 4 students use their understanding of equivalent fractions to compare fractions with different numerators and different denominators.[4.NF.2] For example, to compare $\frac{5}{8}$ and $\frac{7}{12}$ they rewrite both fractions as

$$\frac{60}{96}\left(= \frac{12 \times 5}{12 \times 8}\right) \quad \text{and} \quad \frac{56}{96}\left(= \frac{7 \times 8}{12 \times 8}\right)$$

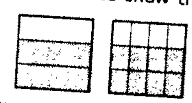
Because $\frac{60}{96}$ and $\frac{56}{96}$ have the same denominator, students can compare them using Grade 3 methods and see that $\frac{56}{96}$ is smaller, so

$$\frac{7}{12} < \frac{5}{8}.$$

Students also reason using benchmarks such as $\frac{1}{2}$ and 1. For example, they see that $\frac{7}{8} < \frac{13}{12}$ because $\frac{7}{8}$ is less than 1 (and is

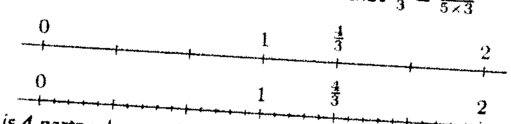
4.NF.1 Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

Using an area model to show that $\frac{2}{3} = \frac{4 \times 2}{4 \times 3}$



The whole is the square, measured by area. On the left it is divided horizontally into 3 rectangles of equal area, and the shaded region is 2 of these and so represents $\frac{2}{3}$. On the right it is divided into 4×3 small rectangles of equal area, and the shaded area comprises 4×2 of these, and so it represents $\frac{4 \times 2}{4 \times 3}$.

Using the number line to show that $\frac{4}{3} = \frac{5 \times 4}{5 \times 3}$



$\frac{4}{3}$ is 4 parts when each part is $\frac{1}{3}$, and we want to see that this is also 5×4 parts when each part is $\frac{1}{5 \times 3}$. Divide each of the intervals of length $\frac{1}{3}$ into 5 parts of equal length. There are 5×3 parts of equal length in the unit interval, and $\frac{4}{3}$ is 5×4 of these. Therefore $\frac{4}{3} = \frac{5 \times 4}{5 \times 3} = \frac{20}{15}$.

4.NF.2 Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Students also compute sums of whole numbers and fractions, by representing the whole number as an equivalent fraction with the same denominator as the fraction, e.g.

$$7\frac{1}{5} = 7 + \frac{1}{5} = \frac{35}{5} + \frac{1}{5} = \frac{36}{5}.$$

Students use this method to add mixed numbers with like denominators.• Converting a mixed number to a fraction should not be viewed as a separate technique to be learned by rote, but simply as a case of fraction addition.

Similarly, converting an improper fraction to a mixed number is a matter of decomposing the fraction into a sum of a whole number and a number less than 1.[4.NF.3b] Students can draw on their knowledge from Grade 3 of whole numbers as fractions. For example, knowing that $1 = \frac{3}{3}$, they see

$$\frac{5}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3} = 1\frac{2}{3}.$$

Repeated reasoning with examples that gain in complexity leads to a general method involving the Grade 4 NBT skill of finding quotients and remainders.[4.NBT.6] For example,

$$\frac{47}{6} = \frac{(7 \times 6) + 5}{6} = \frac{7 \times 6}{6} + \frac{5}{6} = 7 + \frac{5}{6} = 7\frac{5}{6}.$$

When solving word problems students learn to attend carefully to the underlying unit quantities. In order to formulate an equation of the form $A + B = C$ or $A - B = C$ for a word problem, the numbers A, B, and C must all refer to the same (or equivalent) wholes or unit amounts.[4.NF.3d] For example, students understand that the problem

> Bill had $\frac{2}{3}$ of a cup of juice. He drank $\frac{1}{2}$ of his juice. How much juice did Bill have left?

cannot be solved by subtracting $\frac{2}{3} - \frac{1}{2}$ because the $\frac{2}{3}$ refers to a cup of juice, but the $\frac{1}{2}$ refers to the amount of juice that Bill had, and not to a cup of juice. Similarly, in solving

> If $\frac{1}{4}$ of a garden is planted with daffodils, $\frac{1}{3}$ with tulips, and the rest with vegetables, what fraction of the garden is planted with flowers?

students understand that the sum $\frac{1}{3} + \frac{1}{4}$ tells them the fraction of the garden that was planted with flowers, but not the number of flowers that were planted.

Multiplication of a fraction by a whole number Previously in Grade 3, students learned that 3×7 can be represented as the number of objects in 3 groups of 7 objects, and write this as $7 + 7 + 7$. Grade 4 students apply this understanding to fractions, seeing

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \quad \text{as} \quad 5 \times \frac{1}{3}.$$

• A mixed number is a whole number plus a fraction smaller than 1, written without the + sign, e.g. $5\frac{3}{4}$ means $5 + \frac{3}{4}$ and $7\frac{1}{5}$ means $7 + \frac{1}{5}$.

4.NF.3b Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

4.NF.3d Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

d Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

$2.7 = \frac{27}{10}$. Students use their ability to convert fractions to reason that $2.70 = 2.7$ because

$$2.70 = \frac{270}{100} = \frac{10 \times 27}{10 \times 10} = \frac{27}{10} = 2.7.$$

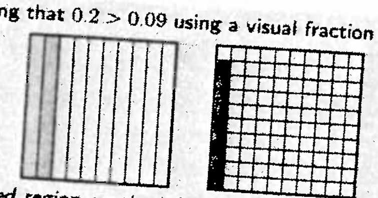
Students compare decimals using the meaning of a decimal as a fraction, making sure to compare fractions with the same denominator. For example, to compare 0.2 and 0.09, students think of them as 0.20 and 0.09 and see that $0.20 > 0.09$ because[4.NF.7]

$$\frac{20}{100} > \frac{9}{100}.$$

The argument using the meaning of a decimal as a fraction generalizes to work with decimals in Grade 5 that have more than two digits, whereas the argument using a visual fraction model, shown in the margin, does not. So it is useful for Grade 4 students to see such reasoning.

4.NF.7 Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model.

Seeing that $0.2 > 0.09$ using a visual fraction model



The shaded region on the left shows 0.2 of the unit square, since it is two parts when the square is divided int 10 parts of equal area. The shaded region on the right shows 0.09 of the unit square, since it is 9 parts when the unit is divided into 100 parts of equal area.

Multiplying and dividing fractions In Grade 4 students connected fractions with addition and multiplication, understanding that

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 5 \times \frac{1}{3}.$$

In Grade 5, they connect fractions with division, understanding that

$$5 \div 3 = \frac{5}{3},$$

or, more generally, $\frac{a}{b} = a \div b$ for whole numbers a and b, with b not equal to zero.[5.NF.3] They can explain this by working with their understanding of division as equal sharing (see figure in margin). They also create story contexts to represent problems involving division of whole numbers. For example, they see that

> If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get?

can be solved in two ways. First, they might partition each pound among the 9 people, so that each person gets $50 \times \frac{1}{9} = \frac{50}{9}$ pounds. Second, they might use the equation $9 \times 5 = 45$ to see that each person can be given 5 pounds, with 5 pounds remaining. Partitioning the remainder gives $5\frac{5}{9}$ pounds for each person.

Students have, since Grade 1, been using language such as "third of" to describe one part when a whole is partitioned into three parts. With their new understanding of the connection between fractions and division, students now see that $\frac{5}{3}$ is one third of 5, which leads to the meaning of multiplication by a unit fraction:

$$\frac{1}{3} \times 5 = \frac{5}{3}.$$

This in turn extends to multiplication of any quantity by a fraction.[5.NF.4a] Just as

$\frac{1}{3} \times 5$ is one part when 5 is partitioned into 3 parts,

so

$\frac{4}{3} \times 5$ is 4 parts when 5 is partitioned into 3 parts.

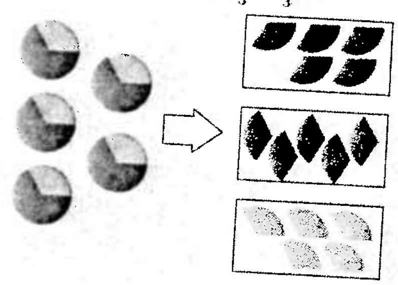
Using this understanding of multiplication by a fraction, students develop the general formula for the product of two fractions,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd},$$

for whole numbers a, b, c, d, with b, d not zero. Grade 5 students need not express the formula in this general algebraic form, but rather reason out many examples using fraction strips and number line diagrams.

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

How to share 5 objects equally among 3 shares:
$5 \div 3 = 5 \times \frac{1}{3} = \frac{5}{3}$

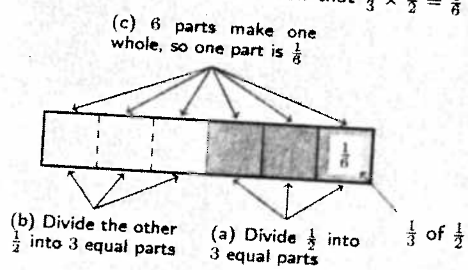


If you divide 5 objects equally among 3 shares, each of the 5 objects should contribute $\frac{1}{3}$ of itself to each share. Thus each share consists of 5 pieces, each of which is $\frac{1}{3}$ of an object, and so each share is $5 \times \frac{1}{3} = \frac{5}{3}$ of an object.

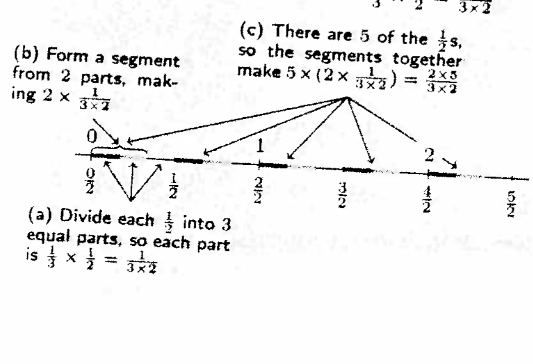
5.NF.4a Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

a Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.

Using a fraction strip to show that $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$



Using a number line to show that $\frac{2}{3} \times \frac{5}{2} = \frac{2 \times 5}{3 \times 2}$



The understanding of multiplication as scaling is an important opportunity for students to reasonin abstractly (MP2). Previous work with multiplication by whole numbers enables students to see multiplication by numbers bigger than 1 as producing a larger quantity, as when a recipe is doubled, for example. Grade 5 work with multiplying by unit fractions, and interpreting fractions in terms of division, enables students to see that multiplying a quantity by a number smaller than 1 produces a smaller quantity, as when the budget of a large state university is multiplied by $\frac{1}{2}$, for example.[5.NF.5b]

The special case of multiplying by 1, which leaves a quantity unchanged, can be related to fraction equivalence by expressing 1 as $\frac{n}{n}$, as explained on page 6.

5.NF.5b Interpret multiplication as scaling (resizing), by:

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.